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Rotational Relativity Theory

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The constancy of the spin of the photon was recently shown to lead to a new Lorentz-type transformation that relates the energy, rotational velocity, moment of inertia, and angular momentum, where rotational invariance was the basis of the theory instead of the ordinary linear invariance of special relativity. In this paper the new group of transformations is shown to lead naturally to a special theory of relativity whose basic metric has an $R \times S^3$ topology rather than the familiar Minkowskian metric. Predictions by the theory are shown to be highly supported by experiment.

In 1905 Einstein published his famous special theory of relativity, revising incorrect physical concepts and fixing our notion of space and time (Einstein, 1905). The theory is based on the two postulates: (1) The speed of light is independent of the source's motion, and (2) the laws of physics are the same in all inertial systems. A deeper understanding of the theory was subsequently given by Minkowski, who showed, among other things, that the Lorentz transformation is nothing but a "rotation" in the four dimensions of space and time (Minkowski, 1909).

However, Einstein was not satisfied with some important aspects of his theory, the most noticeable of which is the inability of special relativity to answer Mach's question why inertial systems are physically distinguished from other systems (Einstein, 1979). And Einstein conceded that he has no answer to this question (Einstein, 1979).

Light, on the other hand, has another important property (in addition to its constant speed), namely, it has a constant intrinsic angular momentum (spin). In a recent paper (Carmeli, 1984) it was shown that the constancy of the spin of the photon leads to a new Lorentz-type transformation that deals with rotational motion and invariance, just as the constancy of the

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speed of light leads to the ordinary Lorentz transformation dealing with linear motion and invariance in the special theory of relativity. The concept of the spin, discovered 20 years after the advent of special relativity (Uhlenbeck and Goudsmit, 1925, 1926), goes along with the theory but is not included in its two postulates. Invariance under rotations with constant angular velocities follows to be intimately related to the constancy of the spin of the photon (Carmeli, 1984). One then deals with bodies (coordinate systems) having constant angular velocities relative to each other and hence Newton's law of inertia for a rigid body (Euler's equations) is valid. In this paper the theory is completed into a theory of relativity that deals with rotational motion, thus showing that inertial systems are not preferred from others as both Mach and Einstein expected.

The proposed theory is, in fact, a special theory of relativity defined on a metric having $R \times S^3$ topology instead of the familiar Minkowskian metric of ordinary special relativity. Here R is the open timelike real line (describing time) and S^3 is the three-sphere (describing rotations), expressed by the group SU_2 which is parametrized by the Euler angles, and the whole space-time is simply connected. The group of transformations obtained (given explicitly in the sequel) is a Lie group and leaves the line element

$$d\tau^2 = dt^2 - \gamma^{-2}[(d\Theta^1)^2 + (d\Theta^2)^2 + (d\Theta^3)^2] \tag{1}$$

invariant under all linear and homogeneous coordinate transformations. Here γ is a constant and $d\Theta^k$, $k = 1, 2, 3$, are the 1-forms

$$\begin{aligned} d\Theta^1 &= \sin\theta \sin\psi \, d\phi + \cos\psi \, d\theta \\ d\Theta^2 &= \sin\theta \cos\psi \, d\phi - \sin\psi \, d\theta \\ d\Theta^3 &= \cos\theta \, d\phi + d\psi \end{aligned} \tag{2}$$

where ϕ, θ, ψ are the Euler angles. $d\Theta^k$ are analogous to the Cartesian differentials dx^k in special relativity. The group of transformations obtained is for rotations with constant angular velocities while the Lorentz group is for translations with constant linear velocities. In both cases they describe "rotations" in four dimensions, as Minkowski discovered, with fixed "angles." The comparable line element to (1) in special relativity is the familiar Minkowskian line element

$$d\tau^2 = dt^2 - c^{-2}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{3}$$

The constant γ appearing in equation (1) is not a universal constant but is characteristic to every particle in nature and is the maximum angular velocity the particle can have (exact definition will be given in the sequel). Also, the infinitesimal quantities $d\Theta^k$ are not differentials of some angles Θ^k but are, mathematically speaking, *imperfect differentials.*

We are now in a position to put forward our postulates for the theory:

(1) The laws of physics are the same in all rotationally unaccelerated systems (bodies) having constant angular velocities relative to each other.

(2) The line element (1) is invariant.

Before we give explicitly the transformation leaving the line element (1) invariant we wish to give another way of looking at the theory, now from the point of view of the general relativity theory. If one starts from the general-relativistic line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (invariant under the infinite coordinate transformations) and lets gravitation have no dynamical role, then one obtains the Minkowskian line element (3), and the group of transformations leaving this line element invariant is the ordinary Lorentz group. In our case if one starts from the general-relativistic homogeneous space metric decribing a finite body $ds^2 = g_{\mu\nu}d\Theta^\mu d\Theta^\nu$ (also invariant under the infinite coordinate transformations with $d\Theta^0 = dt$), and lets gravitation again have no dynamical role, one then obtains the rigid-body line element (1) which is the most comparable to the continuum-physics Minkowskian line element (3), and the group of transformations leaving invariant this line element (1) now is not the Lorentz group anymore but the group given in this paper.

To derive the new Lorentz-type transformation one best confines himself to the two-dimensional "rotation" with a fixed "angle" in space and time. One easily sees that when $\phi = \text{const}$ and $\theta = \text{const}$, then the line element (1) reduces to

$$d\tau^2 = dt^2 - \gamma^{-2}\, d\psi^2 = \text{inv.} \tag{4}$$

corresponding to the line element

$$d\tau^2 = dt^2 - c^{-2}\, dx^2 = \text{inv.} \tag{5}$$

of special relativity, where x is one of the Cartesian coordinates. Then for a fixed "rotation," equation (4) yields the transformation

$$dt' = \frac{dt - \Omega\, d\psi/\gamma^2}{(1-\Omega^2/\gamma^2)^{1/2}}, \qquad d\psi' = \frac{d\psi - \Omega\, dt}{(1-\Omega^2/\gamma^2)^{1/2}} \tag{6}$$

where Ω, with $0 \le \Omega < \gamma$, is the ordinary (three-dimensional) angular velocity $(= d\psi/dt)$, in analogy to the Lorentz transformation obtained from equation (5)

$$dt' = \frac{dt - v\, dx/c^2}{(1-v^2/c^2)^{1/2}}, \qquad dx' = \frac{dx - v\, dt}{(1-v^2/c^2)^{1/2}} \tag{7}$$

where v, with $0 \le v < c$, is the ordinary (three-dimensional) linear velocity $(= dx/dt)$. Multiplying now the line element (4) by $I_0^2\gamma^4/d\tau^2$ and making simple identifications, one then obtains the energy formula

$$E^2 - \gamma^2 J^2 = I_0^2\gamma^4 = E_0^2, \qquad J = I_0\omega = I\Omega \tag{8}$$

where J is the angular momentum, I is the moment of inertia (the subscript 0 refers to the "rest" frame), E is the energy, and ω is the relativistic angular velocity. Equation (8) is analogous to that of special relativity,

$$E^2 - c^2p^2 = m_0^2c^4 = E_0^2, \qquad p = m_0u = mv \tag{9}$$

where p, m, E, and u are the linear momentum, mass, energy, and the relativistic velocity, and the subscript 0 again refers to the "rest" frame. (Comparison of the dynamical variables in linear and rotational motion is given in Table I.) It will be noted that $E = I\gamma^2$ just as $E = mc^2$ in special relativity.

The energy formula (8) can also be written in the form

$$E^2 - l^2\omega^2 = I_0^2\gamma^4 = E_0^2, \qquad l = I_0\gamma \tag{10}$$

just as one can rewrite equation (9) in the form

$$E^2 - k^2u^2 = m_0^2c^4 = E_0^2, \qquad k = m_0c \tag{11}$$

where $\omega = d\psi/d\tau$ and $u = dx/d\tau$. Notice that equations (9) and (10) are valid for any particle, including the photon ($l_{\text{ph}} = \hbar$), whereas equations (8) and (11) are good for finite-mass particles only. From equation (8) one obtains

$$E' = \frac{E - \Omega J}{(1 - \Omega^2/\gamma^2)^{1/2}}, \qquad J' = \frac{J - \Omega E/\gamma^2}{(1 - \Omega^2/\gamma^2)^{1/2}} \tag{12}$$

just as for the energy and linear momentum in special relativity,

$$E' = \frac{E - vp}{(1 - v^2/c^2)^{1/2}}, \qquad p' = \frac{p - vE/c^2}{(1 - v^2/c^2)^{1/2}} \tag{13}$$

To the first order in Ω/γ, equations (12) yield $E' \simeq E - \Omega J$, which is the familiar Routh transformation (Bengtsson and Frauendorf, 1979).

Table I

Dynamical variable (description): Ordinary special relativity	Rotational special relativity
$p = m_0u = mv$ (linear momentum)	$J = I_0\omega = I\Omega$ (angular momentum)
m_0 (rest mass)	I_0 (rest moment of inertia)
$m = m_0/(1 - v^2/c^2)^{1/2}$ (variable mass)	$I = I_0/(1 - \Omega^2/\gamma^2)^{1/2}$ (variable moment of inertia)
$u = v/(1 - v^2/c^2)^{1/2}$ (relativistic velocity)	$\omega = \Omega/(1 - \Omega^2/\gamma^2)^{1/2}$ (relativistic angular velocity)
$p = q/(1 - v^2/c^2)^{1/2}$ (linear momentum)	$J = j/(1 - \Omega^2/\gamma^2)^{1/2}$ (angular momentum)
$q = m_0v$ (nonrelativistic linear momentum)	$j = I_0\Omega$ (nonrelativistic angular momentum)
$k = m_0c$ (characteristic linear momentum)	$l = I_0\gamma$ (characteristic angular momentum)
$v/c = q/k$ (relativistic factor)	$\Omega/\gamma = j/l$ (angular relativistic factor)
$v = dx/dt$ (ordinary velocity)	$\Omega = d\psi/dt$ (ordinary angular velocity)
$u = dx/d\tau$ (relativistic velocity)	$\omega = d\psi/d\tau$ (relativistic angular velocity)
$E = mc^2$ (energy of a particle)	$E = I\gamma^2$ (energy of a particle)
$v_{\text{ph}} = c$ (velocity of photon)	$l_{\text{ph}} = \hbar$ (angular momentum of photon)
$E_{\text{ph}} = cp$ (energy of photon)	$E_{\text{ph}} = \hbar\omega$ (energy of photon)

Predictions by the theory can now be made and compared to experiment. From equation (8) one gets

$$E = E_0/(1-\Omega^2/\gamma^2)^{1/2} \tag{14}$$

which, to the fourth power in Ω/γ, gives

$$E_0 = E - \tfrac{1}{2}I_0\Omega^2 - \tfrac{3}{8}I_0\Omega^4/\gamma^2 \tag{15}$$

analogously to the special-relativistic formula

$$E_0 = E - \tfrac{1}{2}m_0v^2 - \tfrac{3}{8}m_0v^4/c^2 \tag{16}$$

One can compare E_0 with the reference Routhian (Bengtsson and Frauendorf, 1979),

$$E_g = \tfrac{1}{8}\hbar^2/\mathscr{I}_0 - \tfrac{1}{2}\mathscr{I}_0\Omega^2 - \tfrac{1}{4}\mathscr{I}_1\Omega^4 \tag{17}$$

One sees that the two formulas have identical structure and thus we can relate the Harris parameters $\mathscr{I}_0$ and $\mathscr{I}_1$ to the rest moment of inertia I_0 and the maximum rotational frequency γ. One obtains $I_0 = \mathscr{I}_0$ and $\gamma = (3\mathscr{I}_0/2\mathscr{I}_1)^{1/2}$. The difference between equations (15) and (17) is only in the Ω-independent first terms on the right-hand sides, where in equation (15) it is the total energy, whereas in equation (17) it is taken as the nonrelativistic energy $J^2/2\mathscr{I}_0$ with $J = \hbar/2$. In Table II we list the predicted maximum

Table II

Nucleus	Predicted critical rotational frequency $\gamma(10^{20}$ rot/sec$)^a$	Maximum frequency measured^b $\Omega(10^{20}$ rot/sec$)^c$	Rotationally relativistic factor Ω/γ
^{157}Er	1.21	0.92	0.76
^{158}Er	1.40	1.16	0.83
^{159}Er	1.50	0.92	0.61
^{161}Er	1.26	0.92	0.73
^{163}Er	1.54	0.92	0.60
^{164}Er	1.76	0.97	0.55
^{165}Er	1.66	0.92	0.55
^{159}Yb	1.08	0.97	0.90
^{161}Yb	1.36	0.97	0.71
^{163}Yb	1.50	0.87	0.58
^{165}Yb	1.68	0.97	0.58
^{167}Yb	1.43	0.70	0.49
^{169}Yb	1.68	0.92	0.55
^{165}Tm	1.54	0.82	0.53

[a]The predicted critical rotational frequency γ is given by $\gamma = (3\mathscr{I}_0/2\mathscr{I}_1)^{1/2}$.

[b]Bengtsson and Frauendorf (1979).

[c]The Harris parameters $\mathscr{I}_0$ and $\mathscr{I}_1$, as well as the maximum measured rotational frequency Ω, are taken from figures 2, 3, 5, 7, 9, and 12 of Bengtsson and Frauendorf (1979).

rotational frequencies γ for the nuclei whose parameters $\mathcal{J}_0$ and $\mathcal{J}_1$ are given in Bengtsson and Frauendorf (1979) and the maximum measured rotational velocities Ω for each nucleus. One sees that the ratio Ω/γ can be as high as 0.9 (for ^{159}Yb) but does not exceed the limit 1.

We are now in a position to give an exact definition to the constant γ, which can easily be done as in determining the speed of light c in special relativity. From equations (8) and (9) one gets $\gamma^2 = (E/J)(\partial E/\partial J)$ and $c^2 = (E/p)(\partial E/\partial p)$ or $\gamma = [\partial E/\partial J]_{J\to\infty}$ and $c = [\partial E/\partial p]_{p\to\infty}$.

In conclusion, and because of the group property of the transformations obtained, it appears that angular velocities cannot be added indefinitely but have a law of addition of the form

$$\Omega = \frac{\Omega_1 + \Omega_2}{1 + \Omega_1\Omega_2/\gamma^2} \tag{18}$$

This prediction, along with that of the energy formula $E = I\gamma^2$, should be a challenge to experimentalists to prove or disprove. The $R \times S^3$ wave equations, invariant under the transformation presented in this paper, were given elsewhere (Carmeli, 1985; Carmeli and Malin, 1985a,b).

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